

# SYMPLECTIC EMBEDDINGS OF POLYDISKS

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ABSTRACT. If  $P$  is a polydisk with radii  $R_1 \leq \dots \leq R_n$  and  $P'$  is a polydisk with radii  $R'_1 \leq \dots \leq R'_n$ , then we prove that  $P$  symplectically embeds in  $P'$  provided that  $C(n)R_1 \leq R'_1$  and  $C(n)R_1 \dots R_n \leq R'_1 \dots R'_n$ . Up to a constant factor, these conditions are optimal.

## 1. INTRODUCTION

In this paper, we study when it is possible to symplectically embed one polydisk into another. The volume gives one obstruction to finding an embedding, and Gromov's non-squeezing theorem [2] gives a second obstruction. We prove that, up to a constant factor, these are the only obstructions.

We will work in  $\mathbb{R}^{2n}$  with coordinates  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$ , and with the standard symplectic form  $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ . We let  $P$  denote the polydisk  $B^2(R_1) \times \dots \times B^2(R_n)$ , defined by the inequalities  $x_1^2 + y_1^2 < R_1^2, \dots, x_n^2 + y_n^2 < R_n^2$ . We assume that the radii are ordered so that  $R_1 \leq \dots \leq R_n$ . Similarly, we let  $P'$  denote the polydisk  $B^2(R'_1) \times \dots \times B^2(R'_n)$ , with  $R'_1 \leq \dots \leq R'_n$ . If  $P$  symplectically embeds in  $P'$ , then the conservation of volume implies  $R_1 \dots R_n \leq R'_1 \dots R'_n$  and the non-squeezing theorem implies  $R_1 \leq R'_1$ .

**Theorem 1.** *There is a dimensional constant  $C(n)$  so that the following holds. If  $C(n)R_1 \leq R'_1$  and  $C(n)R_1 \dots R_n \leq R'_1 \dots R'_n$  then  $P$  symplectically embeds into  $P'$ .*

There is a linear embedding from  $P$  into  $P'$  roughly if and only if  $R_k \lesssim R'_k$  for every  $k$ . The first interesting non-linear embeddings were constructed by Traynor in [9]. Using Traynor's methods, one can embed  $P$  into  $P'$  roughly if and only if  $R_1 \dots R_k \lesssim R'_1 \dots R'_k$  for every  $k$  between 1 and  $n$ . Until this paper, it was not known whether the conditions  $R_1 \dots R_k \lesssim R'_1 \dots R'_k$  were necessary for  $1 < k < n$ .

The problem of determining exactly when one polydisk symplectically embeds in another looks very difficult. The known results are discussed in detail in [6], and many other embedding problems are discussed in [8].

### Intermediate capacities

Symplectic embeddings are closely related to symplectic capacities. Modulo details, a (generalized) symplectic capacity is a function  $c$  that assigns a number to each open set in  $\mathbb{R}^{2n}$  in such a way that if  $U$  symplectically embeds in  $V$  then  $c(U) \leq c(V)$ . The function  $c$  should also scale in a reasonable way if we scale the set  $U$ , and it obeys some other mild conditions. (See [6] for all details.) The volume is a trivial example of a generalized capacity. The first non-trivial capacity comes from Gromov's proof of the non-squeezing theorem. In [1], Ekeland and Hofer constructed an infinite sequence of symplectic capacities. Since then, several authors have constructed new capacities. The review paper [6] gives a survey of

the field. Finding new capacities is an important topic in symplectic geometry. If Theorem 1 had been false, then there would have been other generalized capacities, significantly different from the known ones.

Many capacities involve the 2-dimensional area of some object, such as a pseudo-holomorphic curve. On the other hand, the  $2n$ -dimensional volume of a region is a generalized capacity. In [5], Hofer asked whether there are intermediate capacities that involve  $2k$ -dimensional volumes for  $1 < k < n$ . Hofer defined an intermediate capacity of dimension  $k$  to be a generalized capacity  $c$  with the following properties:

1.  $c[B^{2k}(1) \times \mathbb{R}^{2n-2k}] < \infty$ .
2.  $c[B^{2k-2}(1) \times \mathbb{R}^{2n-2k+2}] = \infty$ .

For  $k = 1$ , these conditions are satisfied by many interesting examples as described in [6]. For  $k = n$ , these conditions are satisfied by the volume. But for the intermediate range, there are no examples known of such a capacity. If the capacity  $c$  is also reasonably continuous, then it would satisfy the following slightly stronger properties.

- 1'.  $\lim_{R \rightarrow \infty} c[B^{2k}(1) \times B^{2n-2k}(R)] < \infty$ .
- 2'.  $\lim_{R \rightarrow \infty} c[B^{2k-2}(1) \times B^{2n-2k+2}(R)] = \infty$ .

If  $k$  lies in the intermediate range  $1 < k < n$ , Theorem 1 implies that there are no reasonably continuous capacities of dimension  $k$ .

### Expanding embeddings and symplectic embeddings

Let  $X$  denote an open set in  $\mathbb{R}^n$ . The cotangent bundle  $T^*X$  is a symplectic manifold which we can think of as the set of all pairs  $(x, y)$  with  $x \in X$  and  $y \in \mathbb{R}^n$ , equipped with the standard symplectic structure  $\omega$ . We write  $U^*X$  to denote the unit ball cotangent bundle, consisting of all pairs  $(x, y)$  with  $x \in X$  and  $|y| < 1$ .

Any smooth embedding  $I$  from  $X$  to  $X'$  induces a symplectic embedding from  $T^*X$  into  $T^*X'$ . We say that  $I$  is an expanding embedding if  $I$  increases the length of every curve in  $X$ , or equivalently if  $|dI(v)| \geq |v|$  for every tangent vector  $v$  in  $TX$ . If  $I$  is an expanding embedding, then it induces a symplectic embedding from  $U^*X$  into  $U^*X'$ .

If there is no expanding embedding from  $X$  to  $X'$ , then it's interesting to know whether there is a symplectic embedding from  $U^*X$  into  $U^*X'$ . If there is no symplectic embedding, then we can say that the symplectic geometry has remembered that  $X$  does not fit into  $X'$ . How much Riemannian geometry is remembered by the symplectic geometry of the unit ball cotangent bundle?

Our results give a pretty good understanding of this question when  $X$  and  $X'$  are rectangles. Let  $X$  be the rectangle  $[0, L_1] \times \dots \times [0, L_n]$  with the convention  $L_1 \leq \dots \leq L_n$ . Let  $X'$  be the rectangle  $[0, L'_1] \times \dots \times [0, L'_n]$  with  $L'_1 \leq \dots \leq L'_n$ . The following proposition from [4] describes when it is possible to find an expanding embedding from  $X$  into  $X'$ .

**Proposition.** *Up to a constant factor  $C(n)$ , there is an expanding embedding from  $X$  into  $X'$  if and only if  $L_1 \dots L_k \lesssim L'_1 \dots L'_k$  for each  $k$  in the range  $1 \leq k \leq n$ .*

Using Theorem 1, we get the following information about symplectic embeddings of unit ball cotangent bundles.

**Proposition.** *Up to a constant factor  $C(n)$  there is a symplectic embedding from  $U^*X$  into  $U^*X'$  if and only if  $L_1 \lesssim L'_1$  and  $L_1 \dots L_n \lesssim L'_1 \dots L'_n$ .*

Hence the symplectic geometry remembers the two obstructions  $L_1 \lesssim L'_1$  and  $L_1 \dots L_n \lesssim L'_1 \dots L'_n$  but forgets the other  $n-2$  obstructions. In dimension  $n = 2$ , the symplectic geometry is roughly equivalent to the geometry of expanding embeddings (at least for rectangles). In dimension  $n \geq 3$ , the symplectic geometry becomes more flexible than the geometry of expanding embeddings.

### A physical analogy

Traynor's work proves Theorem 1 in the case  $n = 2$ , so the first new embedding happens in the case  $n = 3$ . Here is a typical example. Suppose that  $P$  is equal to  $B^2(\delta) \times B^2(1) \times B^2(1)$  where  $\delta > 0$  is an arbitrarily small number. Suppose that  $P'$  is equal to  $B^2(2\delta) \times B^2(10\delta) \times \mathbb{R}^2$ . In this paper, we will construct a symplectic embedding from  $P$  into  $P'$ . Notice that in this case,  $R_1 R_2$  is much larger than  $R'_1 R'_2$ . I would like to call this embedding the catalyst map because of the following analogy with physics.

We think of a physical system with three degrees of freedom, described by  $(x_1, y_1)$ , by  $(x_2, y_2)$ , and by  $(x_3, y_3)$ . For example, the system could consist of three particles each moving in 1-dimensional space:  $x_i$  would denote the position of particle  $i$ , and  $y_i$  would denote the momentum of particle  $i$ . At the initial time, we might know that the system lies in the polydisk  $P$ . In other words, we have detailed knowledge of the state of particle 1, and medium knowledge of the states of particles 2 and 3. We would like to get more control over particle 2, and we are willing to lose control of particle 3. At the end, we will probably discard particle 3 as exhaust. The system will evolve by a Hamiltonian diffeomorphism, and for the purposes of this discussion let us suppose that we can apply to the system any symplectic embedding. If we just work with particles 2 and 3, Gromov's non-squeezing theorem limits our ability to do what we want to do: we cannot even symplectically embed  $B^2(1) \times B^2(1)$  into  $B^2(1/2) \times \mathbb{R}^2$ . We introduce a new, highly organized component, particle 1, and then evolve the system by the catalyst map so that it lands in  $P'$ . By letting all three particles interact, we are able to improve our knowledge of particle 2. Particle 1 plays the role of a catalyst: it comes out almost unchanged, but with its help, particles 2 and 3 have interacted in a way they could not have done on their own. At the end of the interaction, our knowledge of the catalyst has degraded by a factor of 2, but we have improved our knowledge of particle 2 by an arbitrary factor.

(It would be interesting to know if any real-world systems behave in a way similar to the catalyst map. There is an important caveat about trying to relate the non-squeezing theorem to practical problems in physics: a symplectic embedding can map an arbitrary fraction of the volume of the unit ball into a thin cylinder. Given this caveat, I don't see any reason to think physical systems would behave like the catalyst map.)

### Outline

We let  $\Sigma$  denote a surface of genus 1 with one boundary component, equipped with a symplectic form of area 1. The key step in our proof is the following lemma.

**Main Lemma.** *For any radius  $R$ , the ball  $B^4(R)$  symplectically embeds into  $\Sigma \times \mathbb{R}^2$ .*

We prove the main lemma in Section 2. The lemma builds on work of Polterovich, who constructed a similar embedding using a closed torus instead of  $\Sigma$ . To prove

the main lemma, we modify Polterovich's construction in order to avoid a small disk in the torus.

In Section 3, we use the main lemma to prove Theorem 1. We also give two corollaries.

**Corollary 1.** *For any radius  $R$  and any  $\epsilon > 0$ , there is an immersion from  $B^4(R)$  into the standard cylinder  $B^2(1) \times \mathbb{R}^2$  so that each point in the range has at most two preimages and so that the set of points with two preimages has volume less than  $\epsilon$ .*

This corollary shows that the non-squeezing theorem cannot be weakened to allow immersions. The other corollary of the main lemma is a non-embedding result which builds on the non-squeezing theorem. Let  $\Sigma(\epsilon)$  denote a surface with genus 1 and one boundary component equipped with a symplectic form of area  $\epsilon^2$ .

**Corollary 2.** *Suppose that  $\Sigma(\epsilon) \times B^2(R)$  symplectically embeds in the standard cylinder  $B^2(1) \times \mathbb{R}^2$ . No matter what the value of  $\epsilon$ ,  $R \leq 1$ .*

In an appendix, we review the connection between expanding embeddings and symplectic embeddings. From this point of view, we construct some symplectic embeddings of polydisks similar to those of Traynor in [9].

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## 2. THE MAIN LEMMA

Let  $\Sigma$  denote a surface of genus 1 with one boundary component. We equip  $\Sigma$  with a symplectic form of area 1. The key step in our proof is a symplectic embedding in four dimensions described in the following lemma. We let  $B^{2n}(R)$  denote the ball of radius  $R$  in  $\mathbb{R}^{2n}$  equipped with the standard symplectic form.

**Main Lemma.** *For any radius  $R \geq 1/3$ , there is a symplectic embedding from  $B^4(R)$  into  $\Sigma \times B^2(10R^2)$ .*

This main lemma builds on a result of Polterovich. Let  $T^2(1)$  denote a torus equipped with a symplectic form of area 1.

**Lemma.** (Polterovich) *For any radius  $R \geq 1/3$ , there is a symplectic embedding from  $B^4(R)$  into  $T^2(1) \times B^2(10R^2)$ .*

Our lemma is a stronger version of Polterovich's. We can think of  $\Sigma$  as  $T^2(1)$  with a point  $p$  removed. To prove our lemma, we have to modify Polterovich's embedding so that its image avoids  $\{p\} \times \mathbb{R}^2$ .

Before turning to the proof, let us indicate how to use the main lemma. Because of our lemma, we can embed  $B^2(1/10) \times B^4(R)$  into  $B^2(1/10) \times \Sigma \times \mathbb{R}^2$ . In the next section, we will give a straightforward symplectic embedding from the product  $B^2(1/10) \times \Sigma$  into  $B^2(1/5) \times B^2(1)$ . Combining these two embeddings, we get an embedding from  $B^2(1/10) \times B^4(R)$  into  $B^2(1/5) \times B^2(1) \times \mathbb{R}^2$ . Up to scaling, this last embedding is the catalyst map described in the introduction. Notice that we cannot use Polterovich's lemma in this argument because  $T^2(1)$  does not symplectically embed into  $\mathbb{R}^4$ .

*Proof of Polterovich's lemma:* The embedding is given by a linear map. We write  $B$  as an abbreviation of  $B^4(R)$ . Let  $V$  denote an oriented 2-dimensional

subspace of  $\mathbb{R}^4$  chosen so that the integral of  $\omega$  over the oriented disk  $B \cap V$  is equal to  $\pi/9$ . (We can find such a subspace by continuity.) Now there is a linear symplectomorphism  $L$  which maps planes parallel to  $V$  to planes parallel to the  $x_1 - y_1$  plane, and that maps disks parallel to  $V$  to disks. The image  $L(B)$  is an ellipsoid, and the intersection of  $L(B)$  with any plane parallel to the  $x_1 - y_1$  plane is equal to a disk of radius at most  $1/3$ . The situation is illustrated in Figure 1.

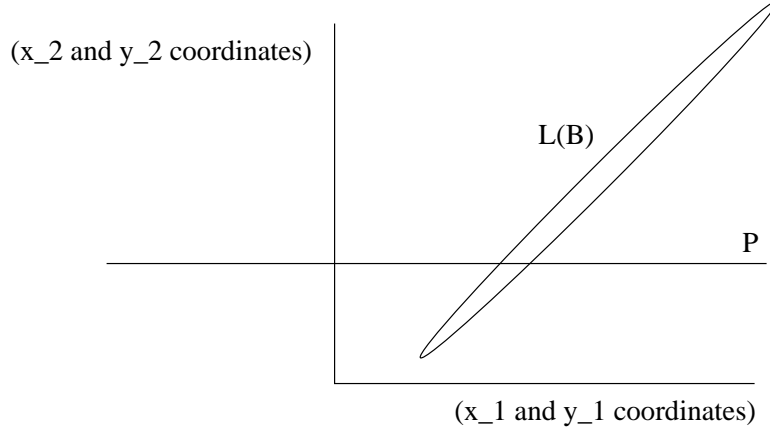


Figure 1. The ellipsoid  $L(B)$  and a 2-plane  $P$ .

(The plane  $P$  in the figure is a 2-plane parallel to the  $x_1 - y_1$  plane. In the figure, the plane  $P$  meets  $L(B)$  in a small disk. Nevertheless, the projection from  $L(B)$  to the  $x_1 - y_1$  plane is large.)

We define  $Q : \mathbb{R}^4 \rightarrow T^2(1) \times \mathbb{R}^2$  to be the quotient map modding out by the integer lattice generated by  $x_1$  and  $y_1$ . The quotient map  $Q$  is symplectic, and  $Q$  restricted to  $L(B)$  is an embedding. To see this, we just intersect  $L(B)$  with any plane parallel to the  $x_1 - y_1$  axis, and look at the quotient map from the intersection to  $T^2$ . Since the intersection is an open disk of radius at most  $1/3$ , the quotient map is injective. Hence  $Q \circ L$  is a symplectic embedding from  $B(R)$  into  $T^2(1) \times \mathbb{R}^2$ .

If we project  $L(B)$  onto the  $x_2 - y_2$  plane, we get an ellipse with area on the order of  $R^4$ . We can choose the linear map  $L$  so that this ellipse is a disk. Next we estimate the area of this disk more carefully. Let us denote its radius by  $S$ . Let  $\pi_2$  denote the projection from  $\mathbb{R}^4$  onto the  $x_2 - y_2$  plane. The point  $\pi_2[L(0)]$  is the center  $c$  of the disk. Hence the preimage  $\pi_2^{-1}(c)$  is a disk of area  $\pi/9$ . Now if  $p$  is any point in the concentric disk of half the radius, then by convexity  $\pi_2^{-1}(p)$  contains a disk of area  $\pi/36$ . Hence the volume of  $L(B)$  is at least  $(\pi/36)\pi(S/2)^2 = (1/144)\pi^2 S^2$ . On the other hand the volume of  $L(B)$  is the same as the volume of  $B$ , which is equal to  $(1/2)\pi^2 R^4$ . Hence  $S < (72)^{1/2} R^2 < 10R^2$ . This finishes the proof of Polterovich's lemma.

*Proof.* Now we turn to the proof of the main lemma. We will proceed by modifying Polterovich's embedding. As before, we write  $B$  to abbreviate  $B^4(R)$ , and we let  $L$  be the linear map constructed above.

Now we outline our strategy. We let  $\Psi$  denote a symplectomorphism of the  $x_1 - y_1$ -plane which we will have to choose carefully later on. We write  $\tilde{\Psi}$  to denote the product of  $\Psi$  with the identity, which is a symplectomorphism of  $\mathbb{R}^4$ . We will

follow the plan of Polterovich's proof above except that we will use the non-linear symplectomorphism  $\bar{\Psi} \circ L$  in place of the linear symplectomorphism  $L$ .

Our embedding will be the composition  $Q \circ \bar{\Psi} \circ L$ , which is automatically a symplectic immersion from  $B^4(R)$  into  $T^2(1) \times \mathbb{R}^2$ . We let 0 denote the point of  $T^2(1)$  corresponding to the integer vectors in the  $x_1 - y_1$  plane, and we identify  $\Sigma$  with  $T^2(1) - \{0\}$ . To prove our lemma, we have to choose  $\Psi$  so that  $Q \circ \bar{\Psi} \circ L(B)$  lands inside of  $\Sigma \times \mathbb{R}^2$ , and so that  $Q$  restricted to  $\bar{\Psi} \circ L(B)$  is an embedding.

We will choose the map  $\Psi$  to obey the two properties below.

We let  $\pi$  denote the projection from  $\mathbb{R}^4$  onto the  $x_1 - y_1$  plane. We choose a number  $\rho$  large enough so that  $\pi(L(B))$  lies in the disk of radius  $\rho$  around the origin.

Property 1. The map  $\Psi$  takes the disk of radius  $\rho$  into the complement of all integer lattice points.

Let  $S$  be a subset of the  $x_1 - y_1$  plane. We say that  $S$  is aperiodic if the difference of any two points in  $S$  is never a non-trivial integer vector.

Property 2. If  $D$  denotes any disk in the  $x_1 - y_1$  plane of radius  $1/3$ , then  $\Psi(D)$  is aperiodic.

Now we check that if  $\Psi$  obeys Properties 1 and 2, then  $Q \circ \bar{\Psi} \circ L$  embeds  $B^4(R)$  into  $\Sigma \times B^2(10R^2)$ . First of all, we have to check that  $Q \circ \bar{\Psi} \circ L(B)$  lands inside of  $\Sigma \times B^2(10R^2)$ . We know that  $L(B)$  lands in  $B^2(\rho) \times B^2(10R^2)$ . By Property 1, we see that  $\bar{\Psi} \circ L(B)$  lands in  $(\mathbb{R}^2 - \mathbb{Z}^2) \times B^2(10R^2)$ . And so  $Q \circ \bar{\Psi} \circ L(B)$  lands in  $\Sigma \times B^2(10R^2)$ .

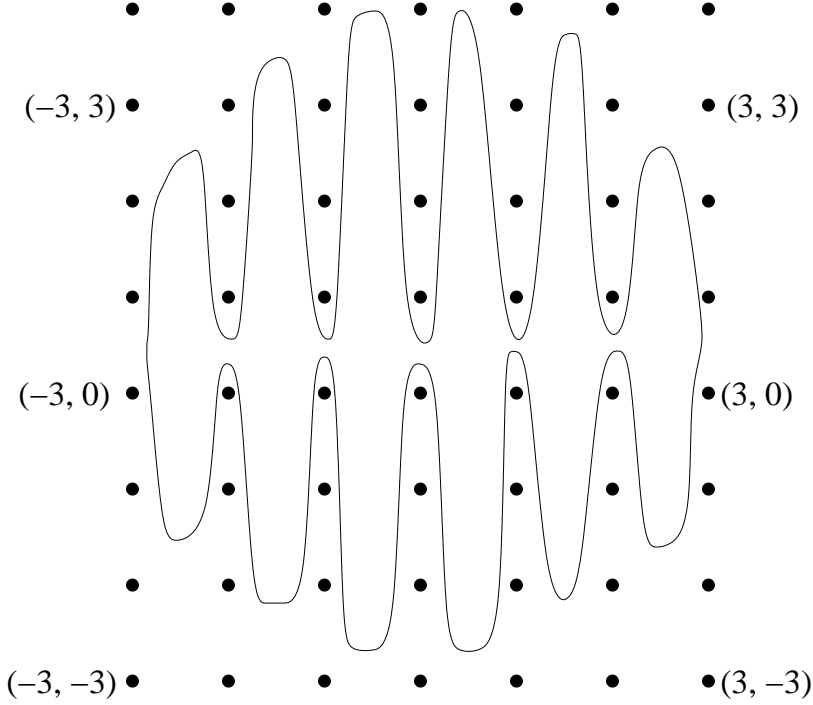
Second, we have to check that  $Q \circ \bar{\Psi} \circ L$  is an embedding from  $B$ . Let  $p, q$  be points in  $B$ , and suppose that  $Q \circ \bar{\Psi} \circ L(p) = Q \circ \bar{\Psi} \circ L(q)$ . It follows that  $\bar{\Psi} \circ L(p) = \bar{\Psi} \circ L(q) + (m, n, 0, 0)$  for some integers  $m$  and  $n$ . In particular we see that  $\bar{\Psi} \circ L(p)$  and  $\bar{\Psi} \circ L(q)$  have the same  $x_2, y_2$  coordinates. Since  $\bar{\Psi}$  doesn't change  $x_2, y_2$  coordinates, it follows that  $L(p)$  and  $L(q)$  have the same  $x_2, y_2$  coordinates. Let  $W$  denote the 2-plane of all points in  $\mathbb{R}^4$  with the same  $x_2$  and  $y_2$  coordinates as  $L(p)$  and  $L(q)$ . The intersection  $W \cap L(B)$  lies in a disk  $D$  of radius  $1/3$ . In particular  $L(p)$  and  $L(q)$  both lie in  $D$ . We can think of  $x_1$  and  $y_1$  as coordinates on this plane  $W$ , and so we can define  $\Psi(D) \subset W$ . Now  $\bar{\Psi} \circ L(p)$  and  $\bar{\Psi} \circ L(q)$  both lie in  $\Psi(D)$ . By Property 2, we know that  $\Psi(D)$  is aperiodic. On the other hand, we established above that  $\bar{\Psi} \circ L(p) = \bar{\Psi} \circ L(q) + (m, n, 0, 0)$ . Therefore,  $m$  and  $n$  are zero,  $p$  is equal to  $q$ , and  $Q \circ \bar{\Psi} \circ L$  does embed  $B$  into  $\Sigma \times B^2(10R^2)$  as claimed.

It remains to construct the map  $\Psi$  with the two properties above. As a tool for constructing  $\Psi$ , we define a diffeomorphism  $\Phi$  of the  $x_1$ -axis. The diffeomorphism  $\Phi$  maps each integer point to itself. It is periodic with period 1. The derivative  $d\Phi$  is always at least  $9/10$ . At each integer point  $d\Phi = 100\rho$ . Finally, the displacement  $|\Phi(x) - x|$  is at most  $10^{-4}$  for every  $x$ . Such a diffeomorphism is easy to find.

Now we define  $\Psi$  by the following formula.

$$\Psi(x, y) = (\Phi(x), (1/2) + [d\Phi(x)]^{-1}y).$$

To give some sense of this map, we sketch the image of the ball of radius 3 under the map  $\Psi$ .

Figure 2. The image of a ball under the map  $\Psi$ 

The map  $\Psi$  is clearly a diffeomorphism. Next we check that it is area preserving by computing its Jacobian. We let  $\Psi_1$  be the x-coordinate of  $\Psi$  and  $\Psi_2$  be the y-coordinate. We compute some derivatives

$$\partial\Psi_1/\partial x = d\Phi(x).$$

$$\partial\Psi_1/\partial y = 0.$$

$$\partial\Psi_2/\partial y = [d\Phi(x)]^{-1}.$$

It follows that the determinant of  $d\Psi$  is equal to 1. Hence  $\Psi$  is a symplectomorphism.

Now we check that  $\Psi$  obeys Properties 1 and 2.

Proof of Property 1. Suppose that  $(x, y)$  is in the disk of radius  $\rho$ . We have to check that  $\Psi(x, y)$  is not an integer lattice point. Suppose that  $\Psi(x, y)$  has x-coordinate equal to an integer. In this case,  $\Phi(x)$  is an integer. By the definition of  $\Phi$  it follows that  $x$  is an integer, and so  $d\Phi(x) = 100\rho$ . But the norm  $|y|$  is at most  $\rho$ . Therefore, the y-coordinate of  $\Psi(x, y)$  is between  $(1/2) - (1/100)$  and  $(1/2) + (1/100)$ . In particular, the y-coordinate is not an integer.

Proof of Property 2. Suppose that  $D$  is a disk of radius  $1/3$ . We have to check that  $\Psi(D)$  is aperiodic. Let  $p$  and  $q$  be two points in  $D$ . We have to check that  $\Psi(p) - \Psi(q)$  is not a non-trivial integer lattice point. We suppose that  $\Psi(p) - \Psi(q)$  is an integer lattice point and prove that  $p = q$ .

From the definition of  $\Phi$ , we know that the displacement  $|\Phi(x) - x|$  is at most  $10^{-4}$ . Therefore, the x-coordinate of  $\Psi(p)$  agrees with the x-coordinate of  $p$  up to an error of  $10^{-4}$ . Similarly for  $q$ . Therefore,  $|\Psi_1(p) - \Psi_1(q)| \leq (2/3) + 2 \cdot 10^{-4}$ . Since the difference  $\Psi_1(p) - \Psi_1(q)$  is an integer, we conclude that the difference is

zero. But the x-coordinate  $\Psi_1(p)$  only depends on the x-coordinate of  $p$ , and so the x-coordinates of  $p$  and  $q$  are equal. Let  $x_0$  be the x-coordinate of each point.

Now the y-coordinate of  $\Psi(p)$  is  $1/2 + [d\Phi(x_0)]^{-1}y(p)$  and the y-coordinate of  $\Psi(q)$  is  $1/2 + [d\Phi(x_0)]^{-1}y(q)$ . Hence their difference is  $[d\Phi(x_0)]^{-1}(y(p) - y(q))$ . But by the definition of  $\Phi$ ,  $d\Phi$  is at least  $9/10$ . Therefore, the difference has norm at most  $(10/9)|y(p) - y(q)| \leq (10/9)(2/3) < 1$ . Since we assumed the difference is an integer, the difference is zero. Then it follows that  $y(p) = y(q)$  and finally that  $p = q$ . This finishes the proof of Property 2 and hence the proof of the main lemma.  $\square$

### 3. CONSEQUENCES OF THE MAIN LEMMA

In this section, we use the main lemma to prove Theorem 1 and afterwards give some other minor consequences. To prove Theorem 1 we need one more lemma, which gives us a way to embed  $B^2(W) \times \Sigma$  into a polydisk.

**Lemma 3.1.** *If  $W \leq 1/10$ , then  $B^2(W) \times \Sigma$  symplectically embeds in  $B^2(2W) \times B^2(1)$ .*

*Proof.* We begin by choosing a symplectic immersion  $I$  of  $\Sigma$  into  $B^2(1)$ . We choose the immersion so that the image of  $\Sigma$  meets the unit square  $[-1/2, 1/2]^2$  in a particularly simple form. Namely, the image contains the strip  $S = [-1/2, 1/2] \times [-W, W]$  and the strip  $S' = [-W, W] \times [-1/2, 1/2]$ . Other than these two strips, the immersion does not hit the square  $[-1/2, 1/2]^2$ . We can arrange that the immersion  $I$  is an embedding except for the overlap of these two strips. The immersion is illustrated in Figure 3.

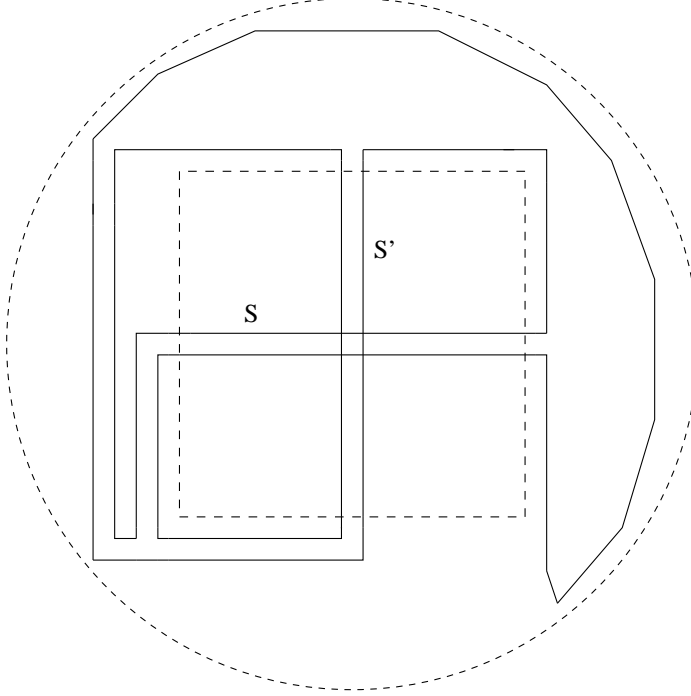


Figure 3. An immersed surface.



The dotted circle on the outside of the figure is the unit circle. The dotted square is the unit square  $[-1/2, 1/2]^2$ . The solid lines are the boundary of the immersed copy of  $\Sigma$ . The horizontal strip  $S$  and the vertical strip  $S'$  are labelled in the figure.

Now the map  $I' = I \times id$  gives a symplectic immersion from  $\Sigma \times B^2(W)$  into  $B^2(1) \times \mathbb{R}^2$ . We will modify this immersion to make it a symplectic embedding. Roughly, we are going to lift the strip  $S \times B^2(W)$  in the  $y_2$  direction a distance  $2W$  in order to push it just over the strip  $S' \times B^2(W)$ .

We will modify  $I'$  on the region  $S \times B^2(W)$  using a Hamiltonian flow. We use coordinates  $x_1, y_1$  on  $B^2(1)$  and coordinates  $x_2, y_2$  on  $\mathbb{R}^2$ . We use the Hamiltonian  $H = \Psi(x_1)x_2$ , where  $\Psi$  is a bump function, equal to 1 on  $[-1/6, 1/6]$ , equal to 0 outside of  $[-1/3, 1/3]$ , and with  $|\nabla \Psi| \leq 7$ . We run the Hamiltonian flow for time  $T = 2W$ . This defines a family of symplectic embeddings of  $S \times B^2(W)$  into  $B^2(1) \times \mathbb{R}^2$ . All of these embeddings agree with  $I'$  on a neighborhood of  $\pm 1/2 \times [-W, W] \times B^2(W)$ , so they extend to immersions of  $\Sigma \times B^2(W)$  into  $B^2(1) \times \mathbb{R}^2$ .

We claim that at time  $T = 2W$  we get an embedding of  $\Sigma \times B^2(W)$  into  $B^2(1) \times \mathbb{R}^2$ . We used the Hamiltonian flow above to change the embedding of  $S \times B^2(W)$ . Meanwhile, the embedding of  $(\Sigma - S) \times B^2(W)$  is left unchanged. We now have to check that our new embedding of  $S \times B^2(W)$  is disjoint from the original embedding of  $(\Sigma - S) \times B^2(W)$ .

During this calculation it's helpful to notice that the Hamiltonian flow leaves  $x_1$  and  $x_2$  invariant. As a first step, we check that at time  $T = 2W$ , our embedding of  $S \times B^2(W)$  lands in  $[-1/2, 1/2]^2 \times \mathbb{R}^2$ . Since  $x_1$  doesn't change during the flow, it follows that the  $x_1$ -coordinate of our embedding of  $S \times B^2(W)$  lies in  $[-1/2, 1/2]$ . Next we deal with the  $y_1$ -coordinate. At the initial time  $|y_1| \leq W$ . During the Hamiltonian flow,  $y_1$  changes at the rate  $\Psi'(x_1)x_2$ . The gradient  $\Psi'(x_1)$  has norm at most 7. Also, the  $x_2$ -coordinate doesn't change during the flow and it has norm at most  $W$ . Therefore,  $y_1$  changes at a rate at most  $7W$ . Hence, at time  $T = 2W$ ,  $|y_1| \leq W + 7W(2W)$ . Because  $W \leq 1/10$ , we conclude that at the final time  $|y_1| < 1/2$ .

The only part of  $(\Sigma - S) \times B^2(W)$  that lies in  $[-1/2, 1/2]^2 \times \mathbb{R}^2$  is the other strip  $S' \times B^2(W)$ . We check that the image of  $S \times B^2(W)$  is disjoint from the vertical strip  $S' \times B^2(W)$ . Suppose that  $(x_1, y_1, x_2, y_2)$  lies in the image of our embedding of  $S \times B^2(W)$ . If  $|x_1| > 1/6 > W$ , then this point does not lie in  $S' \times B^2(W)$ . It remains to consider the case that  $|x_1| \leq 1/6$ . We recall that  $x_1$  did not change during the Hamiltonian flow. The derivative of  $y_2$  during the flow was  $\Psi(x_1) = 1$ . Since we ran the flow for time  $2W$ , the value of  $y_2$  increased during the flow by  $2W$ . Since  $y_2$  was initially in  $(-W, W)$ , at time  $T = 2W$  we have  $y_2 > W$ . Hence our point is disjoint from  $S' \times B^2(W)$ . This argument shows that we have defined an embedding from  $\Sigma \times B^2(2W)$  into  $B^2(1) \times \mathbb{R}^2$ .

The final two coordinates of our embedding always have the form of a point in  $B^2(W)$  plus a vector in the positive  $y_2$  direction of length at most  $2W$ . Therefore the image lies in  $B^2(1) \times B^2([0, W], 2W]$ , where the second term denotes the ball of radius  $2W$  around the point  $(0, W)$  in the  $x_2 - y_2$  plane.  $\square$

Combining this lemma with the main lemma we can construct the catalyst map described in the introduction. Let  $R$  denote a large radius. By the main lemma, we can embed  $B^2(1/10) \times B^2(R) \times B^2(R)$  into  $B^2(1/10) \times \Sigma \times B^2(20R^2)$ . Now applying the last lemma, we can embed this shape into  $B^2(1/5) \times B^2(1) \times B^2(20R^2)$ . After

scaling, this map is the catalyst map. Note that the first radius gets bigger by a factor of only 2, while the second radius gets smaller by a factor of  $R$ .

We remark that this construction is not completely explicit. The surface  $\Sigma$  appears in two different ways. First, we take the  $x_1 - y_1$  plane, mod out by the action of  $\mathbb{Z}^2$ , and then remove a small disk or a point. Second, we take the immersed surface in Figure 3 above with the induced symplectic form. These two surfaces are symplectomorphic by Moser's theorem [7]. The catalyst map is the composition of three steps: the map from the main lemma, then a Moser symplectomorphism, and then the map constructed in Lemma 3.1.

To get further embeddings, we need to combine our construction with embeddings coming from Traynor's work. Also, to help keep track of what embeds in what, we adopt the following notation. If  $P$  is a polydisk with radii  $R_1 \leq \dots \leq R_n$ , and if  $C > 0$ , then we write  $CP$  to denote the magnified polydisk with radii  $CR_1 \leq \dots \leq CR_n$ .

**Proposition 1.** (*Traynor*) *There exists a constant  $C > 0$  so that the following holds. Suppose  $P$  is a polydisk with radii  $R_1 \leq R_2$ . Suppose  $1 \leq \lambda \leq (R_2/R_1)^{1/2}$ . Let  $P'$  be the polydisk with  $R'_1 = \lambda R_1$ , and  $R'_2 = R_2/\lambda$ . Then  $P$  symplectically embeds into  $CP'$ .*

(If  $R'_1 = R'_2$ , then this result follows immediately from Theorem 1.3 of [9]. The general case can be proved using the same method. In the appendix, we sketch a proof of this proposition using expanding embeddings.)

By combining these tools, we get the following generalization of the catalyst map.

**Proposition 2.** *There exists a constant  $C > 0$  so that the following holds. Suppose  $P$  is a polydisk with radii  $R_1 \leq R_2 \leq R_3$ . Suppose  $1 \leq \lambda \leq R_2/R_1$ . Let  $P'$  be the polydisk with  $R'_1 = R_1$ ,  $R'_2 = R_2/\lambda$  and  $R'_3 = R_3\lambda$ . Then  $P$  symplectically embeds into  $CP'$ .*

*Proof.* By Proposition 1 (due to Traynor), we know that  $B^2(R_2) \times B^2(R_3)$  symplectically embeds in  $B^4(CR_2^{1/2}R_3^{1/2})$ . Applying the main lemma and a scaling argument, this ball symplectically embeds in  $\Sigma(R'_2) \times B^2(CR'_3)$ , where  $\Sigma(R'_2)$  denotes a surface of genus 1 with one boundary component equipped with a symplectic form of area  $(R'_2)^2$ . Hence  $B^2(R_1) \times B^2(R_2) \times B^2(R_3)$  symplectically embeds into  $B^2(R'_1) \times \Sigma(R'_2) \times B^2(CR'_3)$ . By the last lemma,  $B^2(R'_1) \times \Sigma(R'_2)$  symplectically embeds in  $B^2(CR'_1) \times B^2(CR'_2)$ . Hence  $P$  symplectically embeds in  $CP'$ .  $\square$

Our main theorem follows from combining these two propositions. It requires no new ideas, but the algebra is a bit tedious.

**Theorem 1.** *For each integer  $n$ , there is a constant  $C(n)$  so that the following holds. Let  $P$  and  $P'$  be polydisks of dimension  $2n$ . Suppose that  $R_1 \leq R'_1$  and  $R_1 \dots R_n \leq R'_1 \dots R'_n$ . Then  $P$  symplectically embeds in  $C(n)P'$ .*

*Proof.* Using Proposition 1 repeatedly, we see that  $P$  embeds symplectically in  $C(n)P(1)$ , where  $R(1)_1 = R_1$  and  $R(1)_i = (R_2 \dots R_n)^{\frac{1}{n-1}}$  for  $2 \leq i \leq n$ . Again, using Proposition 1 repeatedly, we see that  $P(1)$  embeds symplectically in  $C(n)P(2)$ , where  $R(2)_1 = R'_1$  and  $R(2)_i = [R_1 \dots R_n / R'_1]^{\frac{1}{n-1}}$ . Finally, using Proposition 2 repeatedly, we see that  $P(2)$  embeds symplectically in  $C(n)P'$ .  $\square$

To end this section, we give two more consequences of the main lemma.

**Corollary 1.** *For any  $R$  and any  $\epsilon$ , there is a symplectic immersion from  $B^4(R)$  into  $B^2(1) \times \mathbb{R}^2$  so that each point in the range has at most two preimages and so the set of points with two preimages has volume at most  $\epsilon$ .*

*Proof.* First we use the main lemma to embed  $B^4(R)$  into  $\Sigma \times B^2(10R^2)$ . Next, we pick a symplectic immersion from  $\Sigma$  into  $B^2(1)$ . We can choose this immersion so that each point in the target has at most two preimages and so that the set of double points has area at most  $\delta$  for any  $\delta > 0$ . (The immersion we need is illustrated in Figure 3 above.) Composing the immersion and the embedding we get a symplectic immersion from  $B^4(R)$  into  $B^2(1) \times B^2(10R^2)$ . It has at worst double points and the set of double points has area at most  $10\delta R^2$ .  $\square$

Lastly we give a non-embedding result. Let  $\Sigma(\epsilon)$  denote a rescaling of  $\Sigma$  with symplectic area  $\epsilon^2$ .

**Corollary 2.** *If  $\Sigma(\epsilon) \times B^2(W)$  symplectically embeds in the cylinder  $B^2(1) \times \mathbb{R}^2$  then  $W \leq 1$ , regardless of  $\epsilon$ .*

*Proof.* Suppose  $\Sigma \times B^2(W)$  symplectically embeds in  $B^2(R) \times \mathbb{R}^2$ . We know that  $B^4(W)$  embeds in  $\Sigma \times \mathbb{R}^2$ , and so  $B^6(W)$  embeds in  $\Sigma \times B^2(W) \times \mathbb{R}^2$  which embeds in  $B^2(R) \times \mathbb{R}^4$ . By Gromov's non-squeezing theorem, we conclude that  $R \geq W$ .

Now if  $\Sigma(\epsilon) \times B^2(W)$  symplectically embeds in  $B^2(1) \times \mathbb{R}^2$ , we can scale the domain and range to symplectically embed  $\Sigma \times B^2(\epsilon^{-1}W)$  into  $B^2(\epsilon^{-1}) \times \mathbb{R}^2$ . By the last paragraph, we get  $\epsilon^{-1}W \leq \epsilon^{-1}$ , and so  $W \leq 1$ .  $\square$

I don't know whether this result is sharp. Using Lemma 3.1., we can symplectically embed  $\Sigma(\epsilon) \times B^2(W)$  into  $B^2(2W) \times \mathbb{R}^2$  for any values of  $\epsilon$  and  $W$ . The construction in Lemma 3.1 can be improved by using a square in the  $x_2 - y_2$  coordinates instead of a disk. With this improvement, we can symplectically embed  $\Sigma(\epsilon) \times B^2(W)$  into  $B^2(\sqrt{2}W) \times \mathbb{R}^2$ . I don't know whether the factor  $\sqrt{2}$  can ever be reduced.

#### 4. APPENDIX: EXPANDING EMBEDDINGS AND SYMPLECTIC EMBEDDINGS

In this section we sketch a proof of Proposition 1 using expanding embeddings of rectangles.

If  $M$  is a smooth manifold, then the cotangent bundle  $T^*M$  has a canonical symplectic structure. If  $M$  is a Riemannian manifold, then we let  $U^*M$  denote the unit ball cotangent bundle. If  $M, N$  are Riemannian manifolds, then an embedding  $I$  from  $M$  to  $N$  is called expanding if for any tangent vector  $v$  in  $TM$ ,  $|dI(v)| \geq |v|$ . An expanding embedding increases the length of every curve. Any embedding from  $M$  to  $N$  induces a symplectic embedding from  $T^*M$  into  $T^*N$ . An expanding embedding from  $M$  to  $N$  induces a symplectic embedding from  $U^*M$  to  $U^*N$ .

Suppose that  $X$  and  $X'$  are two-dimensional rectangles:  $X = [0, L_1] \times [0, L_2]$ , with  $L_1 \leq L_2$  and  $X' = [0, L'_1] \times [0, L'_2]$  with  $L'_1 \leq L'_2$ . If  $L_1 \leq L'_1$  and  $L_1 L_2 \leq L'_1 L'_2$ , then there is an expanding embedding from  $X$  into  $5X'$ . This embedding is illustrated in Figure 4.

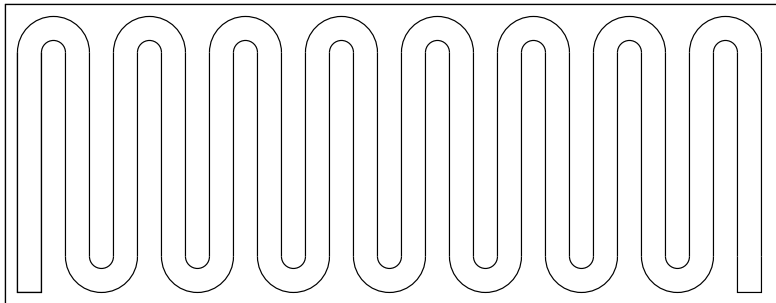


Figure 4. The image of an expanding embedding.

The thicker rectangle on the outside is  $5X'$ . The snake-like shape inside is the image of  $X$  by an expanding embedding.

This expanding embedding induces a symplectic embedding from  $U^*X$  to  $U^*(5X')$ .

The unit ball cotangent bundle  $U^*X$  is not a polydisk, but it's close enough to a polydisk to prove Proposition 1. In particular,  $U^*X$  contains the set of all pairs  $(x, v)$ , where  $x$  is a point in  $X$  and  $v$  is a cotangent vector  $(v_1, v_2)$  with  $|v_1| \leq 2^{-1/2}$  and  $|v_2| \leq 2^{-1/2}$ . The latter shape is a 4-dimensional rectangle and so it is symplectomorphic to a polydisk  $P = B^2(R_1) \times B^2(R_2)$  with radii satisfying  $\pi R_1^2 = 2^{-1/2}L_1$  and  $\pi R_2^2 = 2^{-1/2}L_2$ .

Similarly,  $U^*(5X')$  is contained in the rectangle of points  $(x, v)$ , where  $x$  is a point in  $5X'$  and  $v = (v_1, v_2)$  is a vector with  $|v_i| \leq 1$ . This rectangle is symplectomorphic to a polydisk  $P' = B^2(R'_1) \times B^2(R'_2)$ , with radii satisfying  $\pi(R'_1)^2 = 5L'_1$  and  $\pi(R'_2)^2 = 5L'_2$ .

To summarize,  $P$  symplectically embeds into  $U^*X$ . Then  $U^*X$  symplectically embeds into  $U^*(5X')$ . Finally,  $U^*(5X')$  symplectically embeds into  $P'$ . Hence  $P$  symplectically embeds in  $P'$ . If we begin with  $P$  and  $P'$  obeying  $3R_1 \leq R'_1$  and  $9R_1R_2 \leq R'_1R'_2$ , then a calculation shows that we can symplectically embed  $P$  into  $P'$  using this construction by choosing  $X$  and  $X'$  appropriately. Hence, this construction gives a proof of Proposition 1 with constant  $C = 3$ .

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